

String Theory near a Conifold Singularity

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We demonstrate that type II string theory compactified on a singular Calabi-Yau manifold is related to $c = 1$ string theory compactified at the self-dual radius. We establish this result in two ways. First we show that complex structure deformations of the conifold correspond, on the mirror manifold, to the problem of maps from two dimensional surfaces to S^2 . Using two dimensional QCD we show that this problem is identical to $c = 1$ string theory. We then give an alternative derivation of this correspondence by mapping the theory of complex structure deformations of the conifold to Chern-Simons theory on S^3 . These results, in conjunction with similar results obtained for the compactification of the heterotic string on $K_3 \times T^2$, provide strong evidence in favour of S-duality between type II strings compactified on a Calabi-Yau manifold and the heterotic string on $K_3 \times T^2$.

08/95

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1. Considerable progress has been made in the recent past towards understanding the non-perturbative aspects of string theory. One of the interesting proposals[1], see also [2], is that various different string theories are just different manifestations of presumably a single big picture. Some evidence in this direction has been established using the conjectured string-string duality in six dimensions[3,4] and strong-weak coupling duality (S-duality)[5]. More recently, Kachru and Vafa[6] have given some explicit examples of dual pairs of string theories with $N = 2$ spacetime supersymmetry. They showed that certain Calabi-Yau compactifications of type II string theory are related under S-duality to heterotic string theory compactified on $K_3 \times T^2$.

It has been known for quite some time[7] that the leading terms in the expansion of the prepotential for type II strings compactified on a Calabi-Yau manifold exhibit logarithmic singularities. Strominger[8] has suggested a beautiful interpretation of these singularities. He proposed that conifold singularities in a Calabi-Yau manifold can be understood in terms of extremal black holes becoming massless. Taking these massless fields into account, the transition across the boundary between the moduli space of topologically distinct Calabi-Yau manifolds becomes smooth[9]. As a result different classical vacua of string theory form a connected web. This may eventually help shed some light on understanding the non-perturbative ground state of string theory. It is therefore important to understand the universal features of the conifold singularity in Calabi-Yau manifolds.

It was shown by Vafa[10] that the one loop free energy of the effective field theory of type II string near a conifold has universal behaviour and is given by

$$F_1 = -\frac{1}{12} \log \mu, \quad (1)$$

where μ is the mass of the extremal black hole which becomes massless at the conifold singularity. Following this one loop computation, it was suggested by Ghoshal and Vafa[11] that the conifold singularity in Calabi-Yau threefolds indeed has universal behaviour and is given by $c = 1$ string theory at the self-dual radius. They observe that the leading order behaviour (up to two loops in the string coupling constant) of the free energy of the conifold theory matches with the perturbative expansion of the free energy of the $c = 1$ string at the self-dual radius.

Antoniadis et al. in their recent work[12] have shown that a one loop computation in heterotic string theory, compactified near an enhanced symmetry point in the $K_3 \times T^2$ moduli space, reproduces the full $c = 1$ string free energy. This result is consistent with

the conjectured duality between type II and heterotic strings. To put this conjecture on a more firm footing we need to better understand the relation between type II strings near a conifold and $c = 1$ string theory.

In this paper we will show that the result of [11] can be extended to all orders in string perturbation theory using two different methods.

- i) We employ mirror symmetry to map the type B topological sigma model corresponding to the string theory near the conifold to the A-model on the mirror Calabi-Yau. On this mirror manifold the singularity is resolved into an S^2 , and in order to study the universal behaviour of the string near the singularity we only need to consider mappings from the worldsheet to this S^2 . This is exactly the subject of 2D QCD, and using the relation between 2D QCD and $c = 1$ string theory[13,14,15,16] we find the complete perturbative expansion of the free energy of the conifold theory.
- ii) The theory of complex structure deformations of Calabi-Yau manifolds is given by Kodaira-Spencer theory which was used in [11] to study deformations of the conifold singularity. We show that Kodaira-Spencer theory in the neighbourhood of a conifold singularity reduces to Chern-Simons theory on S^3 . This, combined with the results obtained by Periwal[17], who showed that in the double scaling limit the free energy of Chern-Simons theory on S^3 is the generating function of the Euler characteristics of the moduli spaces of surfaces of genus g , establishes yet another relation between the conifold theory and the $c = 1$ string.

2. Let us briefly review some results obtained earlier on the compactification of type II string theory on the conifold[8,10,11]. The conifold singularity occurs in a Calabi-Yau manifold whenever either a 2-cycle or a 3-cycle shrinks to zero size[7]. This singularity can therefore be resolved by replacing it by either an S^2 (called small resolution) or an S^3 (deformation). As mentioned earlier, the leading term in the free energy expansion of the string compactified on a Calabi-Yau manifold has logarithmic singularities, which can be interpreted as extremal black holes becoming massless[8]. The procedure followed in [11] to desingularize the conifold involves deforming the singularity into an S^3 . Near this singularity the degenerating Calabi-Yau can be described by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu, \tag{2}$$

where x_i are coordinates on \mathbf{C}^4 . As $\mu \rightarrow 0$, the quadric develops a singularity. Nonzero μ corresponds to resolving this singularity into an S^3 . Under a linear redefinition, (2)

possesses a formal similarity with the deformed ground ring of $c = 1$ string theory at the self-dual radius[18,19]. The topological Landau-Ginzburg model corresponding to $c = 1$ string theory is given by the singular superpotential[20,21]

$$W(x) = -\mu x^{-1}. \quad (3)$$

The central charge of this model is equal to that of a Calabi-Yau manifold ($\hat{c} = 3$). By adding quadratic terms to the superpotential, which does not alter the central charge, we can embed this superpotential in an ambient space such that geometrically it is equivalent to a Calabi-Yau[22]. The modified superpotential is

$$W(x) = -\mu x^{-1} + x_1^2 + x_2^2 + x_3^2 + x_4^2. \quad (4)$$

The $W = 0$ locus of (4) satisfies the Calabi-Yau condition and corresponds to the deformed conifold.

The S^3 deformation of the conical singularity corresponds to complex structure deformations of the Calabi-Yau and hence is described by a topological sigma model of type B. Following the techniques developed in [23], Ghoshal and Vafa obtain the terms through genus two in the perturbative expansion of the free energy of the string theory near a conifold singularity. They show the equivalence of these terms with the free energy expansion of the $c = 1$ string.

Using these ingredients they show that the behaviour of type II strings near the conifold singularity in a Calabi-Yau manifold is universal and is captured uniquely by $c = 1$ string theory at the self-dual radius.

3. Mirror symmetry tells us that the topological sigma model of type B on a Calabi-Yau manifold is identical to a topological sigma model of type A on the corresponding mirror manifold. Here we consider the A-model on the mirror, which is the appropriate model to study when we choose to resolve the conifold singularity into an S^2 .

Since we are interested in understanding the behaviour of string theory near the conifold, the only thing that is relevant to understanding its behaviour near the singularity is string dynamics on the new S^2 generated by the resolution. String theory near the singularity therefore reduces to the topological sigma model which counts the instanton maps from the worldsheet to S^2 . The maps from two dimensional surfaces to S^2 have been studied in the context of two dimensional QCD. In particular it has been demonstrated that the latter theory in the $1/N$ expansion can be interpreted in terms of sums over maps

between two dimensional manifolds, with the target manifold being the manifold on which the original QCD theory is defined.

Here we will show, following Gross and Matytsin[13], that two dimensional QCD on a cylinder in the $1/N$ expansion gives rise to the free energy of the $c = 1$ string at the self-dual radius. From the above discussion one would infer that the two sphere, instead of the cylinder, is the space on which the QCD should be defined. This situation is quite reminiscent of the relation between matrix models and gauged WZNW models. In this case, the Penner model is related to the $SU(2)/U(1)$ gauged WZNW model at level k , which is analytically continued to $k = -3$ [24]. It can also be thought of as an $SL(2)/U(1)$ model with $k = 3$ [25]. This analytic continuation turns the compact coset manifold $SU(2)/U(1)$ into a non-compact one corresponding to $SL(2)/U(1)$. It would be interesting to understand this phenomenon at a more fundamental level in this context; however, we will proceed here with the cylinder as the target space of the two dimensional QCD.

The partition function for QCD with gauge group $U(N)$ on a cylinder can be computed exactly for any finite N using heat kernel methods and equals[26]

$$Z_N[U_{C_1}, U_{C_2}; A] = \sum_R \chi_R(U_{C_1}) \chi_R(U_{C_2}^\dagger) e^{-\frac{\lambda A}{2N} C_2(R)}, \quad (5)$$

where $\lambda = g^2 N$ is the coupling constant and A is the area of the cylinder. Since the partition function only depends on their product, we will set $\lambda = 1$. $\chi_R(U)$ is the character of the matrix U in the representation R ; $C_2(R)$ is the second Casimir of this representation.

We now introduce the collective variables $\sigma(\theta)$ to denote the eigenvalue distribution of the unitary matrix U in the large N limit. In this collective field theory approach one can then derive the following equation for the classical action

$$\frac{\partial S}{\partial A} = \frac{1}{2} \int_0^{2\pi} d\theta \sigma_1(\theta) \left[\left(\frac{\partial}{\partial \theta} \frac{\delta S}{\delta \sigma_1(\theta)} \right)^2 - \frac{\pi^2}{3} \sigma_1^2(\theta) \right]. \quad (6)$$

This equation, when we interpret A as time, $\sigma_1(\theta)$ as canonical coordinate, and $\Pi(\theta)$ as its conjugate momentum, corresponds to the Hamilton-Jacobi equation for the Das-Jevicki Hamiltonian[27]

$$H[\sigma(\theta), \Pi(\theta)] = \frac{1}{2} \int_0^{2\pi} d\theta \sigma(\theta) \left[\left(\frac{\partial \Pi(\theta)}{\partial \theta} \right)^2 - \frac{\pi^2}{3} \sigma^2(\theta) \right]. \quad (7)$$

This Hamiltonian describes the effective field theory of $c = 1$ string theory. The free energy of this theory is given by [28,29]

$$F[\mu, R] = \frac{1}{2}\mu^2 \log \mu - \frac{1}{24} \left(R + \frac{1}{R} \right) \log \mu + \sum_{g \geq 2} F_g(R) \mu^{2-2g}, \quad (8)$$

where R is the radius of compactification and μ is the renormalized string coupling constant. The functions F_g are invariant under $R \leftrightarrow 1/R$ and for $R = 1$ equal the Euler character of the moduli space of genus g Riemann surfaces. Matching this expression with the result of the one loop computation [10] in the conifold theory given in (1), we find that the conifold corresponds to $c = 1$ string compactified at the self-dual radius ($R = 1$). Hence the expansion of the free energy of the conifold around $\mu = 0$ is equal to the topological expansion of the $c = 1$ string.

4. Complex structure deformations of a Calabi-Yau manifold are described by Kodaira-Spencer theory. This theory has been shown to reduce to Chern-Simons theory [23] in the following situation. Consider analytic continuation of a Calabi-Yau manifold to a six dimensional symplectic manifold which consists of a 3-dimensional base space X and a 3-dimensional internal space Y . The holomorphic and the anti-holomorphic 3-form inherited from the Calabi-Yau manifold become volume forms on X and Y . The action of the Kodaira-Spencer theory then coincides with the action of the Chern-Simons theory on X with the infinite dimensional group of volume preserving diffeomorphisms of Y as its gauge symmetry. Thus for a fixed holomorphic 3-form Ω on a Calabi-Yau manifold, which becomes the volume form of Y , the Kodaira-Spencer action reduces to the Chern-Simons action.

The $1/N$ expansion [30] of Feynman graphs in the Chern-Simons theory [31] is a topological expansion with N^2 being a genus counting parameter. Periwal has shown [17] that the exact free energy of $SU(N)$ Chern-Simons theory at level k can be expanded in powers of $(N + k)^{-2}$. Due to the level-rank duality symmetry of this expansion it is possible to do both strong coupling (N large with k fixed) and weak coupling (k large with N fixed) expansions.

Let us come back to the conifold theory. In case of the B-model the conifold singularity is deformed into an S^3 . Since the base of the cone is $S^3 \times S^2$, near the apex of the cone the deformed geometry is like $S^3 \times R^3$. This can be interpreted as a 6-dimensional symplectic

manifold with the symplectic structure obtained from the Kähler structure of the Calabi-Yau manifold. In the neighbourhood of this S^3 Kodaira-Spencer theory reduces to Chern-Simons theory with Ω -preserving diffeomorphism symmetry. The partition function of the Chern-Simons theory on S^3 [32] is given by

$$Z[S^3] = S_{0,0}, \quad (9)$$

where S_α^β is the modular transformation matrix corresponding to the action of $SL(2, Z)$ on the characters of the $SU(N)$ WZNW model at level k . The partition function for gauge group $SU(N)$ therefore can be written as[33,34,35]

$$Z[S^3; N, k] = (N + k)^{-N/2} \sqrt{\frac{N + k}{N}} \prod_{j=1}^{N-1} \left[2 \sin \left(\frac{j\pi}{N + k} \right) \right]^{N-j}. \quad (10)$$

In the limit $N \rightarrow \infty$, this Chern-Simons theory possesses volume preserving symmetry. Type II string near a conifold singularity is in the strong coupling regime and since the Chern-Simons theory is an appropriate description near a conifold, the strong coupling expansion (N large with k finite) of its free energy should capture the behaviour of string theory near a conifold. Let us define a new variable $x = N/(N + k)$. Then in the scaling limit where $x \rightarrow 1$ and $k = (N + k)(1 - x)$ is held fixed, expansion of the free energy of the Chern-Simons theory on S^3 is given by

$$F[S^3; k] = \frac{1}{2} k^2 \log k - \frac{1}{12} \log k + \sum_{g \geq 2} \chi_g k^{2-2g} + \text{terms analytic in } k, \quad (11)$$

where $\chi_g = B_{2g}/2g(2g - 2)$ is the Euler character of the moduli space of Riemann surfaces of genus g [17]. After identifying k with the cosmological constant μ it is easy to recognize that this is the well known free energy expansion of $c = 1$ string theory. Hence we see that the universal behaviour of the conifold theory is given by $c = 1$ string theory.

5. In this paper we established the relation between the conifold theory and $c = 1$ string theory using two techniques. We first showed that complex structure deformations of the conifold singularity correspond to maps from 2D surfaces to S^2 in the mirror manifold. With the aid of 2D QCD we demonstrated the equivalence of the free energies of the conifold theory and $c = 1$ string theory. We then rederived this relation by studying Chern-Simons theory on S^3 . Our results along with the results of [12] on the heterotic side present a strong case in favour of S-duality in four dimensional $N = 2$ string theories.

We would like to end with an interesting observation. As is well known, $c = 1$ string theory can be represented in terms of free fermions. In this free fermion representation the $\mu \rightarrow 0$ limit corresponds to vanishing Fermi energy and therefore this theory has gapless excitations. It would be interesting to see if these excitations have any relation to the extremal black holes that become massless at the conifold singularity.

Acknowledgements: We would like to thank N. Berkovits, D. Ghoshal, R. Gopakumar, M. Roček, W. Siegel and C. Vafa for valuable discussions.

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